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Scalar particle production in Schwarzschild and Rindler metrics

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Abstract. The procedure used recently by Hawking to demonstrate the creation of massless particles by black holes is applied to the Rindler coordinate system in flat space–time. The result is that an observer who undergoes a uniform acceleration κ apparently sees a fixed surface radiate with a temperature of $\kappa/2\pi$. Some implications of this result are discussed.

1. Introduction

It has long been appreciated that the definition of 'particle' in quantum field theory depends on the coordinate system used to describe space–time. Even in flat space–time it is possible to construct (Rindler 1966, Fulling 1972, 1973) a self-consistent quantization scheme based on Rindler (rather than Minkowski) coordinates which differs from the conventional quantization.

Rindler coordinates may be defined for two-dimensional Minkowski space by

$$z = (x^2 - t^2)^{1/2} \qquad 0 < z < \infty \tag{1}$$

$$v = \tanh^{-1}(t/x), \qquad -\infty < v < \infty. \tag{2}$$

The coordinates (z, v) cover the wedge-shaped region shown in figure 1. Lines of constant z correspond to world lines of observers undergoing uniform acceleration of z^{-1}. The

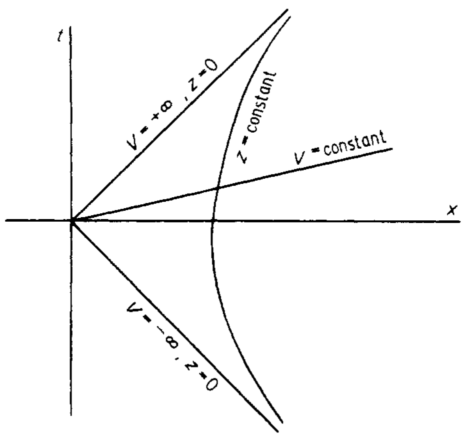

Figure 1.

two asymptotes $z = 0, v = -\infty$ and $z = 0, v = +\infty$ therefore behave as event horizons, and will be referred to as the past and future horizons respectively. The metric tensor of flat space–time in Rindler coordinates is

$$z^2 \, \mathrm{d}v \otimes \mathrm{d}v - \mathrm{d}z \otimes \mathrm{d}z \tag{3}$$

which has a coordinate singularity at $z = 0$.

From the foregoing properties of the Rindler wedge, a close similarity with the static exterior region of a spherically symmetric black hole covered by Schwarzschild coordinates is apparent. An analysis of flat space quantum field theory in Rindler (1966) coordinates may therefore be regarded as a test case for the Schwarzschild system (Fulling 1972, 1973), with the added advantages of conceptual simplicity and the knowledge that the 'correct' quantization scheme based on Minkowski coordinates is always available for comparison.

In this paper a hitherto little suspected property of Rindler quantization of a massless scalar field is discussed. The motivation for this work is the recent result by Hawking (1975) that a steady rate of production of massless particles occurs in the exterior region of a Schwarzschild black hole. This result is somewhat surprising inasmuch as strict particle conservation would normally be expected in this *static* region of space–time (Blum 1973, Davies and Taylor 1974). Hawking's result hinges on the existence of the event horizon in the Schwarzschild system, which divides the solutions of the massless wave equation into two classes—those solutions which manage to propagate from $\mathscr{I}^-$ through the centre of the object and out to $\mathscr{I}^+$, and those which are trapped by the formation of the horizon and do not reach $\mathscr{I}^+$. The sudden variation in the Fourier transform on $\mathscr{I}^-$ due to this division is responsible for the particle production.

It might be expected that a similar situation would arise in the Rindler case. Almost identical properties may be ascribed to the Rindler system by equipping the space with a reflecting wall placed at a distance a to the right of the origin as shown in figure 2. The purpose of this wall is to turn incoming (left-moving) waves into outgoing (right-moving) waves, in the same fashion that incoming waves are changed into outgoing waves by passage through the centre of a collapsing object in the black hole system. A straightforward application of Hawking's argument to this flat space system leads to a Fourier transform which is essentially identical to that of the Schwarzschild case. The upshot of this is that the fixed reflecting wall appears to an accelerating observer to radiate at a

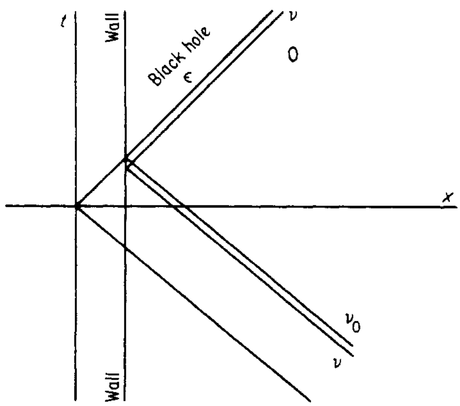

Figure 2.

constant temperature of $(\hbar/2\pi kc)\times$(acceleration) K in cgs units. In geometric units the acceleration may be equated to the surface gravity κ of an object of mass M. In that case the radiation has a temperature of $\kappa/2\pi$ which is exactly the temperature of a black hole with surface gravity κ found by Hawking. Some of the implications of this result are discussed in § 4.

2. The scalar wave equation in Rindler coordinates

The general massless covariant wave equation is taken to be

$$\frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{(-g)}g^{\mu\nu}\partial_\nu\phi] = 0 \tag{4}$$

where $g = \det g_{\mu\nu}$ and $g_{\mu\nu}$ is the metric tensor. In the metric described by (3), equation (4) becomes

$$-\partial_v^2\phi(z, v) = \left(z^2\frac{\partial^2}{\partial z^2}+z\frac{\partial}{\partial z}\right)\phi(z, v)$$

which may readily be solved to yield solutions of the form

$$A_\omega \exp[i\omega(v\pm \ln z)], \tag{5}$$

where ω is a dimensionless constant and changes in the scale of z only affect A_ω to the extent of a factor of modulus unity.

The surfaces v = constant are Cauchy surfaces (for finite v) for the wedge-shaped region covered by the Rindler coordinates, so we may express the field at a general point (z, v) in terms of a complete set of solutions of the type (5) over one of the v = constant surfaces. (The actual details of a consistent quantum field theory for this system have been given by Fulling (1972, 1973) and will not be repeated here.) In particular we may consider the decomposition

$$\phi(z, v) = \sum_\omega (a_\omega f_\omega + a_\omega^\dagger \bar{f}_\omega) \tag{6}$$

on a surface with large negative v. In a quantum theory a_ω, $a_\omega^\dagger$ may be interpreted as annihilation and creation operators for incoming (left-moving) scalar particles respectively, so that we may put

$$f_\omega \sim \omega^{-1/2}\exp[i\omega(v+\ln z)] \tag{7}$$

where the $\omega^{-1/2}$ term has been included because a continuum normalization of the eigenfunctions f_ω is being adopted.

In a region such as the point O of figure 2, contributions to the field disturbance will arise from two sources. First the incoming disturbance discussed above, which will cross the surface $v=\infty$. This surface is a future event horizon for the accelerating observers, so that the region beyond is analogous to a 'black hole'. There will also be disturbances from the left (outgoing) which are caused by reflection at the wall, but which appear to an observer O to have crossed the past event horizon.

In a region near O (v large and positive) we may therefore decompose the field as follows:

$$\phi(z, v) = \sum_\omega (b_\omega g_\omega + b_\omega^\dagger \bar{g}_\omega + c_\omega h_\omega + c_\omega^\dagger \bar{h}_\omega) \tag{8}$$

where g_ω represent solutions which are reflected by the wall and pass out to large z, and h_ω represent solutions which cross the horizon into the 'black hole'. The operators b and c are the respective annihilation and creation operators for particles of these types.

The interesting results come from the former solutions. It is not necessary to consider the detailed form of g_ω in the region of the wall as the particle production effects do not depend on this.

In the region of O we have

$$g_\omega \sim \omega^{-1/2} \exp[i\omega(v - \ln z)] \tag{9}$$

while if we pass backwards in time through the reflection at the wall and out to the region of large negative v, g_ω will be shown *not* to have the form (7) in this region. If we write

$$g_\omega = \int_0^\infty (\alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} \bar{f}_{\omega'})\, d\omega' \tag{10}$$

then $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ are in general nonzero. Thus the vacuum state for incoming particles will not be the vacuum state for an observer near O, who will instead see the production of particles with an expectation value

$$d\omega \int_0^\infty |\beta_{\omega\omega'}|^2\, d\omega' \tag{11}$$

for the range ω to $\omega + d\omega$.

3. Evaluation of $\beta_{\omega\omega'}$

To evaluate the amount of particle production we must first determine the form of g_ω in the region of large negative v. This may easily be achieved by adopting the argument used by Hawking in the black hole case. In figure 2 a null line ν has been drawn parallel to the future event horizon and at a distance (on the diagram) ϵ from it. The equations of null lines in Rindler coordinates are

$$z\, e^{\pm v} = \text{constant} \tag{12}$$

where the constant is the value of z for $v = 0$. The null line ν will have the equation

$$\ln z - v = \ln \sqrt{2} + \ln \epsilon \tag{13}$$

in the region of O. The null lines represent surfaces of constant phase, as may be verified by inserting (13) into (9). The phase for a given ϵ may be chosen as

$$-\omega(\ln \sqrt{2} + \ln \epsilon). \tag{14}$$

As the null line ν approaches the horizon, $\epsilon \to 0$ and the phase (14) diverges.

We may follow ν backwards in time through the reflection and out to the region of large negative v, as shown in figure 2. In this region, as $\epsilon \to 0$, the surfaces of constant phase of g_ω will tend to crowd up faster and faster onto a limiting null line ν_0, after which $g_\omega = 0$, because all later disturbances are 'trapped' by the future event horizon. The situation is therefore almost exactly the same as in the black hole case.

Introducing for convenience an advanced time τ defined by

$$\tau = \ln z + v \tag{15}$$

we may express ϵ in terms of the value of τ. Incoming null lines are given from (12) by

$$z\,\mathrm{e}^{v} = \mathrm{e}^{\tau} \tag{16}$$

so that

$$\epsilon\sqrt{2} = a - \mathrm{e}^{\tau} = \mathrm{e}^{\tau_0} - \mathrm{e}^{\tau} \tag{17}$$

where $\tau_0 = \ln a$. For $(\tau_0 - \tau)$ small and positive, equation (17) may be written

$$\epsilon\sqrt{2} \simeq \mathrm{e}^{\tau_0}(\tau_0 - \tau).$$

It follows that the phase of g_ω in this region is

$$-\omega[\tau_0 + \ln(\tau_0 - \tau)] \tag{18}$$

and that g_ω has the form

$$\begin{array}{ll} (\mathrm{constant})^{\mathrm{i}\omega}\omega^{-1/2}\exp[-\mathrm{i}\omega\ln(\tau_0 - \tau)] & \tau < \tau_0 \\ 0 & \tau > \tau_0. \end{array} \tag{19}$$

The coefficients $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ in equation (10) may be calculated by taking a Fourier transform of g_ω with respect to τ. The result is the same as that given by Hawking (1975):

$$|\alpha_{\omega\omega'}| = \frac{|\Gamma(1 - \mathrm{i}\omega)|}{(\omega\omega')^{1/2}} \tag{20}$$

$$|\beta_{\omega\omega'}| = \frac{\mathrm{e}^{-\pi\omega}|\Gamma(1 - \mathrm{i}\omega)|}{(\omega\omega')^{1/2}}. \tag{21}$$

If equation (21) is substituted into expression (11) for the amount of particle production in the modes ω to $\omega + \mathrm{d}\omega$, it diverges logarithmically. As pointed out by Hawking this result corresponds to a steady rate of particle production over an infinite period of time. This may be demonstrated by constructing wave packets (the reader is referred to Hawking's paper for details). For consider an incoming wave packet approaching the wall at an advanced time $\tau > \tau_0$. In the absence of any potential field (which would cause a back-scattering analogous to the metric back-scattering in the black hole case, and irrelevant to the main argument here) all of this wave packet will cross the future horizon into the 'black hole' region, and will appear to a Rindler observer such as O to have been completely absorbed by the wall. However, O will also see the emission of radiation by the wall, due to the particle production effect, from wave packets which approach the wall for times $\tau < \tau_0$, and then reflect to reach O. To an observer at O all of these outgoing wave packets will appear to have crossed the past horizon (in the absence of the wall) in a fashion identical to that of the passage of the former incoming packet into the 'black hole' region. It follows that there is a fixed ratio of emission to absorption cross section for particles. The ratio, which is essentially determined by equations (20) and (21), turns out to be (Hawking 1975)

$$(\mathrm{e}^{2\pi\omega} - 1)^{-1} \tag{22}$$

which has the form of that for a black body radiator of temperature $(2\pi)^{-1}$ in our units (it does not depend on the value of a).

To convert expression (22) to geometric units note that an interval of proper time along a world line z = constant is given from (3) as $z\,dv$, so that an observer with co-ordinates (z, v) would interpret a wave with time dependence $e^{i\omega v}$ as having a frequency $\omega/z = \omega \times$ (acceleration of observer). To such an observer the wall surface would appear to have a temperature of (acceleration/2π). From the point of view of the observer O, a wall which recedes with acceleration κ in flat space–time acquires a temperature of $\kappa/2\pi$, which is the same temperature as that acquired by the surface of a collapsing star receding with a (proper) acceleration κ.

4. Interpretation of results

It has been emphasized by B S de Witt (1974, unpublished communication) that particle production is expected in Minkowski space containing neutral reflecting surfaces in certain states of motion. In particular such production would be expected from a uniformly accelerated wall. However, the system being quantized here consists of a region of Minkowski space equipped with an unaccelerated wall described by an accelerated coordinate system. The apparent production of particles in this case is somewhat paradoxical, because there is no obvious source of energy for this production. Such emission of radiation is of course absent when the system is quantized in conventional Minkowski coordinates, so that the result demonstrates how the concept of a particle is ill-defined and observer-dependent.

Fulling (1972) has drawn attention to various reasons why one might reject the identification of 'Rindler' particles as defined here with real physical particles. The Rindler coordinates have a singularity at $z = 0$ and the curves of constant z are not geodesics. More seriously, the cordinate system does not cover the entire space–time but only the wedge-shaped region between the two null lines. This objection would not be expected to matter much in the discussion of high energy particles which can be localized in a small region compared to the distance between an observer and his horizon. However, at low energies when the wavelength becomes comparable with the horizon distance, the global properties of the coordinate system will become relevant to the discussion of particles, and the quantization scheme might be expected to become inapplicable. It is significant then that the above calculation (see expression (22)) indicates that particle creation effects are negligible at high frequencies, and only become appreciable when ω^{-1} is comparable to the horizon distance.

At this point it is interesting to recall the close analogy between the Rindler coordinate system and the Schwarzschild system in the exterior region of a spherically symmetric collapsing object. Fulling (1973) has written:

> 'Rindler coordinates are just as appropriate for the description of the region of flat space which they cover as Schwarzschild coordinates are for the study of the space around a massive body outside the radius $r = 2M$. If the theory fails in this test case, it probably must also be rejected for the Schwarzschild metric.'

It might therefore be expected that the present discussion had a superficial bearing on the Hawking result for black holes. Certainly the mathematical details of the particle production rate are very similar. Physically also, the appearance to the accelerated observer of the retreating wall has the same features as that of the retreating surface of a collapsing object to an asymptotic observer. In both cases any intrinsic surface luminosity rapidly fades due to the red shift, while a constant apparent luminosity is superimposed. Quantitatively the expressions for the temperature of this radiation emission are directly analogous, with the acceleration of the wall replacing the surface gravity in Hawking's formula.

Clearly then the definition of particle in the black hole case must also be approached with some caution. In the region of the horizon the quantization scheme may have to be modified.

Indeed Hawking has proposed that a *freely-falling* observer would not see a large amount of particle production near a black hole as he approached the horizon. However, Hawking advances the comparability of long wavelengths with horizon size as evidence for low-frequency particle production, rather than for a break down in the definition of particle in this régime.

In spite of the close similarities between the present calculation and that of Hawking there are some significant differences. For example, in the former case the temperature of the emitted radiation falls like z^{-1} along a null line passing to higher and higher values of z. In the latter case this temperature tends to an asymptotically constant value. The difference can be traced to the fact that Rindler space reproduces the effect of an inverse first-power gravitational field, as opposed to the inverse square behaviour around a collapsing object. Thus there is an additional red shift present in the Rindler case which progressively reduces the temperature of the radiation to zero. Consequently an accelerated observer in the Rindler system will always see the radiation peaked at a wavelength comparable with his horizon distance, however far from the wall he is located. In the black hole case the metric becomes Minkowskian asymptotically and the emitted particles may be localized well away from the horizon. Moreover, Hawking's treatment is based on a Kruskal representation rather than Schwarzschild. Differences such as these make it unclear to what extent the paradoxes of the present result are relevant to Hawking's result.

5. Remarks on exploding black holes

Assuming that black holes do produce real physical particles at the rate predicted by Hawking, there exists a number of important implications for elementary particle physics and cosmology.

Hitherto, it has usually been assumed that the collapse phase may be ignored in considerations of the end states of black holes. This is based on the belief that all collapsing objects whatever their internal constitution, rapidly settle down to one of the standard black hole solutions which are parametrized only by mass, charge and angular momentum. Thus the very term 'black hole' is defended on the grounds that although for an asymptotic observer a collapsing object takes an infinite amount of time to reach the event horizon, and is therefore in principle not a 'collapsed' but always just a 'collapsing' object, nevertheless in practice the properties of the latter grow rapidly like the former as the collapse proceeds. For example, the surface luminosity fades out exponentially on a time scale of order milliseconds for a solar mass object.

Now the production of particles by black holes relies crucially on the early stages of the collapse, when signals may still travel right through the collapsing object and reach an asymptotic observer located on the far side. Indeed, an examination of quantum field theory in the static end state situation only indicates unambiguously that there is *no* particle production (except for that associated with the phenomenon of super-radiance).

In the light of these remarks it seems that Hawking's result obliges us to make a clear distinction between 'black hole' and 'collapsing object' after all. Consequently many of the theorems and properties of black holes discovered in the last few years may well have diminished application in the real world, where one is really only concerned with gravitational collapse.

For example, any collapsing objects with initial masses less than 10^{15} gm would have evaporated away during the lifetime of the universe. Such an object would consist of, say, a collapsing cloud of protons and electrons. In a very short proper time as measured by these collapsing particles, the protons and electrons would disappear, their energy being transformed into photons, neutrinos and gravitons. It is not therefore clear that a horizon will form at all. Consequently any discussion of the situation of these particles in the interior Schwarzschild region would be inappropriate. Whereas it was previously assumed that a particle could cross without incident into the interior Schwarzschild region and then inevitably proceed towards a singularity, it now seems (as far as the author can see) that there is a physical barrier which prevents this. If these conclusions also apply to objects more massive than 10^{15} gm (which might depend on cosmological conditions in the far future), then the whole character of discussion about gravitational collapse seems to be changed.

The consequences of Hawking's result for elementary particle physics are no less drastic. Although it has long been appreciated that the laws of conservation of lepton number, baryon number etc were transcended in black hole physics, at least as far as an asymptotic observer is concerned, this was based on the assumption that a physicist would not have access to the contents of the black hole except by sacrificing himself in so obtaining. However, the physicist is now at liberty to measure quite precisely the baryon number of the object before it collapses, and compare it with the baryon number of the particles emitted, after the object has evaporated. It is hard to see how the conclusion that these conservation laws are actually violated can be avoided.

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